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Singular modes of the electromagnetic field

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Abstract
We show that the mode corresponding to a point of the essential spectrum of the electromagnetic scattering operator is a vector-valued distribution representing the square root of the three-dimensional Dirac's delta function. An explicit expression for this singular mode in terms of the Weyl sequence is provided and analysed. The essential resonance occurs if the permittivity of an object gets close to zero, which is often the case in plasmas and negative-permittivity metamaterials. Such resonance would lead to a perfect localization (confinement) of the electromagnetic field. Simultaneously, however, a portion of electromagnetic energy is removed from the Hilbert space and therefore the whole process may be viewed as absorption.

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1. Introduction

The ability to manipulate the spatial distribution of the electromagnetic field is required in many practical applications. For example, one may wish to create an antenna with a very broad or a very narrow radiation pattern, localize and amplify light or transmit it along a predefined optical path, accelerate charged particles or keep them tight within a fusion chamber. In the presence of matter all possible spatial distributions are encoded in the spatial spectrum of the electromagnetic scattering operator. The most basic field distributions, which are easy to observe in microwave resonators, are called eigenmodes. Manipulation thus amounts to a clever excitation of a particular eigenmode or a combination of those. If almost all electromagnetic energy is carried by one of the modes, then we have a resonance. The concept of electromagnetic resonances and eigenmodes is a natural description of microwave resonators and waveguides [1], microstrip lines [2], and other simple, often, infinite, homogeneous, or periodic structures [3–5]. In a recent paper [6] we have generalized this idea for arbitrary dielectric objects of finite extent. The strongest point of our generalization was the incorporation of the full information about the spatial spectrum of the electromagnetic scattering operator [7], which has both discrete eigenvalues and an essential (continuous) part.

In some applications, e.g. optoelectronics, plasmonics, photonics, near-field optics, the electromagnetic field has to be confined within a very small volume of space (see, e.g., [8–10]). In research on metamaterials, where one strives for a negative permittivity material with vanishing losses, an unusual 'anomalous localized resonance' is also encountered [11, 12]. In both plasmonic and metamaterial cases temporal dispersion leads to values of the dielectric permittivity close to zero. In [6] we show that in the latter case an essential resonance may occur. We also speculated that the modes associated with the essential spectrum of the scattering operator may be highly localized in space. In particular, we argued that the conditions on the essential resonance coincide with the situation where the so-called plasmon is observed at a plasma-dielectric interface. Here we give a rigorous proof that the localization (confinement) of the electromagnetic field is linked to the essential spectrum of the electromagnetic scattering operator and provide the explicit form of the corresponding spatial modes.

Our previous work on the essential spectrum [6, 7] was based on Mikhlin's theory of singular integral operators [13] which does not concern the shape of the modes. In fact, there are very few examples of the analysis of the modes corresponding to the essential spectrum of operators. In quantum mechanics, where the spectral theory is very advanced, the discrete spectrum seems to be of more importance. There the essential spectrum is associated with the unbounded motion of particles [14] and is, therefore, of little interest to physicists. Hence, we had to find some suitable approach to this problem elsewhere.

To recover the structure of the modes corresponding to the electromagnetic essential spectrum we resort here to Weyl's definition of a spectrum (see, e.g., [14, 15]). This definition states that a number λ is in the spectrum of the operator A if and only if there exists a sequence $\{\Psi_n\}$ in the space X such that

$$\|\Psi_n\| = 1 \tag{1}$$

and

$$\lim_{n\to\infty} \|A\Psi_n - \lambda\Psi_n\| = 0. \tag{2}$$

Furthermore, λ is in the *essential spectrum*, if there is a *singular sequence* satisfying (1)–(2), i.e. a sequence which contains no convergent subsequence. If X is a complete Hilbert space, then, obviously, such a singular sequence will not (strongly) converge to any function from X, although it may weakly converge to zero. As we know, sequences that do (strongly) converge to some function on X generate eigenfunctions or *eigenmodes* corresponding to the point spectrum—*eigenvalues*. By analogy we may associate *essential modes* with the essential spectrum. An alternative term, which, perhaps, better reflects the nature and structure of the particular modes obtained here, would be *singular modes*.

The difference between the Mikhlin and the Weyl approaches can be summarized as follows. Mikhlin analysed the mapping properties of singular integral operators, similar to the one we have in three-dimensional electromagnetic scattering. In particular, he was interested in their Fredholm property (the range is closed and the dimensions of the kernel and cokernel are finite). He proved that the *symbol* of singular integral operators provides all the necessary information. Roughly speaking, the operator is Fredholm, if and only if its symbol, which, in general, is a matrix-valued function, is invertible. The symbol of the electromagnetic scattering operator turns out to be a parametric function of the permittivity and is invertible if and only if the dielectric permittivity does not become zero at any point in $\mathbb{R}^3$. In fact, this is the condition on the existence of the solution of the electromagnetic scattering problem. It was rather straightforward to employ this condition and thus the Mikhlin approach to recover the essential spectrum. Indeed, the standard definition of a spectrum considers the residual operator

$$[A - \lambda I]^{-1}, \qquad (3)$$

and looks for points λ where it fails to exist. The essential spectrum is defined as the set of λ's for which the operator $[A - \lambda I]$ fails to be Fredholm. Hence, we only had to derive the symbol of $[A - \lambda I]$ and see if it fails to be invertible for some λ's. In the isotropic electromagnetic case this happens for $\lambda = \varepsilon(\mathbf{x}, \omega)/\varepsilon_0$; thus the essential spectrum is comprised of all values of the relative permittivity.

The Weyl approach takes a different and by far less explored route focusing on the shape of eigenfunctions. Strictly speaking, the original Weyl's definition of the essential spectrum was meant for self-adjoint operators only. The electromagnetic scattering operator is not self-adjoint nor is it even normal. In the modern literature, however, the Weyl definition (1)–(2) corresponds to the *approximate point spectrum* of arbitrary operators. Since we have already derived the essential spectrum in a rigorous way, the Weyl approach to recovering the modes was more of a lucky guess, with a little hint from the theory of pseudospectrum of non-normal operators [16]. In our view, the relation between the Mikhlin and the Weyl approaches to the recovery of the essential spectrum of arbitrary operators requires further analysis.

2. The square root of the delta function

Before going into the details of the electromagnetic case we shall introduce a peculiar function that is later used to generate the essential modes. On the one hand, the sequence of such functions should be singular, as required by the definition mentioned above. On the other hand, the application of (2) to electromagnetics (in L_2-norm) and consistency with the previously obtained rigorous results on the essential spectrum [7] require that the square of this function should behave like the Dirac delta function, i.e., must have the sifting property. Hence, what we need is a *square root of the delta function*.

From time to time the square roots of delta functions appear in the literature. Mostly, though, just as a curious example of a non-convergent sequence, see [17, p 299] and [18, p 81]. A more modern and rigorous approach to such functions is the Colombeau algebra [19], where one studies the products of distributions and encounters *m-singular delta functions*, which are almost identical to what we are after. At present, the main applications of the Colombeau algebra are nonlinear equations (e.g. hydrodynamics, elastodynamics and general relativity), singular shock waves in nonlinear conservation laws and propagation of delta-like waves in linear media with discontinuous parameters. In other words, this algebra is applied whenever a potentially meaningless product of generalized functions is stumbled upon. The present apparent emergence of the Colombeau algebra in the recovery of the essential spectrum seems to be new and can, probably, be generalized. The main difference from the established applications of this algebra is in the use of the fractional powers (roots) of distributions rather than their higher powers (squares, cubes, etc).

Unfortunately, we could not find any explicit derivation of the required distribution in the literature. The available one-dimensional and scalar three-dimensional [14, pp 74–75] examples are of no use to us, since the electromagnetic essential spectrum is a purely three-dimensional phenomenon and our function and its Fourier transform had to have a very special vectorial structure. In view of the potential usefulness of the obtained result in other areas of research, we have decided to devote this entire section to the analysis of the square root of the three-dimensional delta function. The proof of the following theorem is, however, rather technical and can be skipped in the first reading.

Theorem 1. *The vector-valued function*

$$\Psi(\alpha, \mathbf{x}, \mathbf{x}_c) = \left(\frac{2}{3}\right)^{1/2} \pi^{-3/4}\alpha^{5/4}(\mathbf{x}-\mathbf{x}_c) \exp\left(-\frac{\alpha}{2}|\mathbf{x}-\mathbf{x}_c|^2\right), \tag{4}$$

where $\mathbf{x}, \mathbf{x}_c \in \mathbb{R}^3$ *and* $\alpha > 0$*, has the following properties:*

(i) It is normalized in the sense that

$$\|\Psi(\alpha, \mathbf{x}, \mathbf{x}_c)\|_2 = 1. \tag{5}$$

(ii) The sequence of such functions

$$\Psi(\alpha_n, \mathbf{x}, \mathbf{x}_c), \qquad \alpha_n > \alpha_{n-1}, \qquad n = 1, 2, \ldots \tag{6}$$

does not have a convergent subsequence.

(iii) Its Fourier transform is given by

$$\tilde{\Psi}(\alpha, \mathbf{k}, \mathbf{x}_c) = -\mathrm{i}\left(\frac{2}{3}\right)^{1/2} \pi^{-3/4}\alpha^{-5/4}\mathbf{k} \exp\left(-\frac{1}{2\alpha}|\mathbf{k}|^2 - \mathrm{i}\mathbf{k}\cdot\mathbf{x}_c\right). \tag{7}$$

(iv) It is a generator of the square-root of the Dirac delta-function, i.e., with any bounded continuous function $f(\mathbf{x})$ *its square has the sifting property*

$$\lim_{\alpha\to\infty} \int_{\mathbf{x}\in\mathbb{R}^3} f(\mathbf{x})\, |\Psi(\alpha, \mathbf{x}, \mathbf{x}_c)|^2 \,\mathrm{d}\mathbf{x} = f(\mathbf{x}_c). \tag{8}$$

(v) It is orthogonal to bounded vector-valued functions, i.e., for any $|\mathbf{V}(\mathbf{x})| < \infty$, $\mathbf{x} \in \mathbb{R}^3$,

$$\lim_{\alpha\to\infty} \langle \mathbf{V}, \Psi\rangle = \lim_{\alpha\to\infty} \int_{\mathbf{x}\in\mathbb{R}^3} \Psi^{\mathrm{T}}(\alpha, \mathbf{x}, \mathbf{x}_c)\mathbf{V}(\mathbf{x}) \,\mathrm{d}\mathbf{x} = 0. \tag{9}$$

(vi) It is 'invisible' to weakly singular operators with finite spatial support, i.e.

$$\lim_{\alpha\to\infty} \left\| \int_{\mathbf{x}\in D} \frac{\mathbb{K}(\mathbf{x}, \mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^{\beta}} \Psi(\alpha, \mathbf{x}, \mathbf{x}_c) \,\mathrm{d}\mathbf{x} \right\|_2 = 0, \tag{10}$$

where $\mathbb{K}(\mathbf{x}, \mathbf{x}')$, $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^3$ *is a bounded tensor-valued function,* $\beta < 3$*, and the norm is defined over the spatial support* D*.*

Proof. (i) To prove the normalization property we simply compute

$$\begin{aligned}
\|\Psi(\alpha, \mathbf{x}, \mathbf{x}_c)\|_2^2 &= \frac{2}{3}\pi^{-3/2}\alpha^{5/2} \int_{\mathbf{x}\in\mathbb{R}^3} |\mathbf{x}-\mathbf{x}_c|^2 \,\mathrm{e}^{-\alpha|\mathbf{x}-\mathbf{x}_c|^2} \,\mathrm{d}\mathbf{x} \\
&= \frac{2}{3}\pi^{-3/2}\alpha^{5/2} \int_{r=0}^{\infty}\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} r^4 \,\mathrm{e}^{-\alpha r^2} \sin\theta \,\mathrm{d}\varphi \,\mathrm{d}\theta \,\mathrm{d}r \\
&= \frac{2}{3}\pi^{-3/2}\alpha^{5/2} 4\pi \frac{(4-1)!!}{2(2\alpha)^2}\left(\frac{\pi}{\alpha}\right)^{1/2} = 1,
\end{aligned} \tag{11}$$

where we have used the following standard integral:

$$\int_0^{\infty} r^{2n}\,\mathrm{e}^{-pr^2}\,\mathrm{d}r = \frac{(2n-1)!!}{2(2p)^n}\sqrt{\frac{\pi}{p}}, \qquad p > 0, \qquad n = 0, 1, 2, \ldots \tag{12}$$

(ii) Suppose that there is a subsequence $\Psi(\alpha_n, \mathbf{x}, \mathbf{x}_c)$, where $\alpha_n > \alpha_{n-1}, n = 1, 2, \ldots,$ which converges in norm. Then, for any $\epsilon > 0$ there exists N such that for all $m, n > N$ we have

$$\|\Psi(\alpha_m, \mathbf{x}, \mathbf{x}_c) - \Psi(\alpha_n, \mathbf{x}, \mathbf{x}_c)\|^2 \leqslant \epsilon. \tag{13}$$

However, in our case we obtain

$$
\begin{aligned}
\|\Psi(\alpha_m, \mathbf{x}, \mathbf{x}_c) - \Psi(\alpha_n, \mathbf{x}, \mathbf{x}_c)\|^2 &= \int_{\mathbf{x}\in\mathbb{R}^3} |\Psi(\alpha_m, \mathbf{x}, \mathbf{x}_c) - \Psi(\alpha_n, \mathbf{x}, \mathbf{x}_c)|^2 \,\mathrm{d}\mathbf{x} \\
&= \int_{\mathbf{x}\in\mathbb{R}^3} |\Psi(\alpha_m, \mathbf{x}, \mathbf{x}_c)|^2 \,\mathrm{d}\mathbf{x} + \int_{\mathbf{x}\in\mathbb{R}^3} |\Psi(\alpha_n, \mathbf{x}, \mathbf{x}_c)|^2 \,\mathrm{d}\mathbf{x} \\
&\quad - 2\int_{\mathbf{x}\in\mathbb{R}^3} \Psi^T(\alpha_m, \mathbf{x}, \mathbf{x}_c)\Psi(\alpha_n, \mathbf{x}, \mathbf{x}_c) \,\mathrm{d}\mathbf{x} \\
&= 2 - 2^{7/2}\frac{\alpha_m^{5/4}\alpha_n^{5/4}}{(\alpha_m+\alpha_n)^{5/2}} \int_{\mathbf{x}\in\mathbb{R}^3} |\Psi(\alpha_m+\alpha_n, \mathbf{x}, \mathbf{x}_c)|^2 \,\mathrm{d}\mathbf{x} \\
&= 2 - 2^{7/2}\frac{\alpha_m^{5/4}\alpha_n^{5/4}}{(\alpha_m+\alpha_n)^{5/2}}.
\end{aligned}
\tag{14}
$$

If we now fix $n > N$ and let $m > N$ go to infinity, then the last term tends to zero leaving us with a constant (two), which, obviously, cannot be made smaller than an arbitrary $\epsilon > 0$. This proves that there are no convergent subsequences.

(iii) The Fourier transform is obtained by a direct computation as follows:

$$
\begin{aligned}
\tilde{\Psi}(\alpha, \mathbf{k}, \mathbf{x}_c) &= (2\pi)^{-3/2} \int_{\mathbf{x}\in\mathbb{R}^3} \Psi(\alpha, \mathbf{x}, \mathbf{x}_c) \exp(-\mathrm{i}\mathbf{k}\cdot\mathbf{x}) \,\mathrm{d}\mathbf{x} \\
&= (2\pi)^{-3/2}\left(\frac{2}{3}\right)^{1/2} \pi^{-3/4}\alpha^{5/4} \int_{\mathbf{x}\in\mathbb{R}^3} (\mathbf{x}-\mathbf{x}_c)\exp\left(-\frac{\alpha}{2}|\mathbf{x}-\mathbf{x}_c|^2 - \mathrm{i}\mathbf{k}\cdot\mathbf{x}\right)\mathrm{d}\mathbf{x} \\
&= (2\pi)^{-3/2}\left(\frac{2}{3}\right)^{1/2} \pi^{-3/4}\alpha^{5/4} \exp(-\mathrm{i}\mathbf{k}\cdot\mathbf{x}_c) \int_{\mathbf{y}\in\mathbb{R}^3} \mathbf{y}\exp\left(-\frac{\alpha}{2}|\mathbf{y}|^2 - \mathrm{i}\mathbf{k}\cdot\mathbf{y}\right)\mathrm{d}\mathbf{y} \\
&= (2\pi)^{-3/2}\left(\frac{2}{3}\right)^{1/2} \pi^{-3/4}\alpha^{5/4} \exp(-\mathrm{i}\mathbf{k}\cdot\mathbf{x}_c)(\mathrm{i}\nabla_{\mathbf{k}}) \int_{\mathbf{y}\in\mathbb{R}^3} \exp\left(-\frac{\alpha}{2}|\mathbf{y}|^2 - \mathrm{i}\mathbf{k}\cdot\mathbf{y}\right)\mathrm{d}\mathbf{y} \\
&= (2\pi)^{-3/2}\left(\frac{2}{3}\right)^{1/2} \pi^{-3/4}\alpha^{5/4} \exp(-\mathrm{i}\mathbf{k}\cdot\mathbf{x}_c)(\mathrm{i}\nabla_{\mathbf{k}}) \int_{-\infty}^{\infty} \exp\left(-\frac{\alpha}{2}y_1^2 - \mathrm{i}k_1y_1\right)\mathrm{d}y_1 \\
&\quad \times \int_{-\infty}^{\infty} \exp\left(-\frac{\alpha}{2}y_2^2 - \mathrm{i}k_2y_2\right)\mathrm{d}y_2 \times \int_{-\infty}^{\infty} \exp\left(-\frac{\alpha}{2}y_3^2 - \mathrm{i}k_3y_3\right)\mathrm{d}y_3.
\end{aligned}
\tag{15}
$$

Each of the one-dimensional integrals above gives

$$
\begin{aligned}
\int_{-\infty}^{\infty} \exp\left(-\frac{\alpha}{2}y_n^2 - \mathrm{i}k_ny_n\right)\mathrm{d}y_n &= \int_{-\infty}^{\infty} \exp\left[-\frac{\alpha}{2}\left(y_n^2 + \mathrm{i}\frac{2}{\alpha}k_ny_n\right)\right]\mathrm{d}y_n \\
&= \int_{-\infty}^{\infty} \exp\left[-\frac{\alpha}{2}\left(y_n^2 + 2y_n\left(\mathrm{i}\frac{k_n}{\alpha}\right) + \left(\mathrm{i}\frac{k_n}{\alpha}\right)^2 - \left(\mathrm{i}\frac{k_n}{\alpha}\right)^2\right)\right]\mathrm{d}y_n \\
&= \exp\left(-\frac{1}{2\alpha}k_n^2\right)\int_{-\infty}^{\infty} \exp\left[-\frac{\alpha}{2}\left(y_n + \frac{\mathrm{i}}{\alpha}k_n\right)^2\right]\mathrm{d}y_n \\
&= \exp\left(-\frac{1}{2\alpha}k_n^2\right)\int_{-\infty}^{\infty} \exp\left[-\pi\left(\sqrt{\frac{\alpha}{2\pi}}y_n + \frac{\mathrm{i}}{\sqrt{2\pi\alpha}}k_n\right)^2\right]\mathrm{d}y_n \\
&= \left(\frac{2\pi}{\alpha}\right)^{1/2}\exp\left(-\frac{1}{2\alpha}k_n^2\right)\int_{-\infty+\mathrm{i}b}^{\infty+\mathrm{i}b} \exp[-\pi(z+\mathrm{i}b)^2]\mathrm{d}(z+\mathrm{i}b) \\
&= \left(\frac{2\pi}{\alpha}\right)^{1/2}\exp\left(-\frac{1}{2\alpha}k_n^2\right).
\end{aligned}
\tag{16}
$$

Substituting (16) in (15) we arrive at our result:

$$\tilde{\Psi}(\alpha, \mathbf{k}, \mathbf{x}_c) = \mathrm{i}(2\pi)^{-3/2}\left(\frac{2}{3}\right)^{1/2}\pi^{-3/4}\alpha^{5/4}\exp(-\mathrm{i}\mathbf{k}\cdot\mathbf{x}_c)\nabla_{\mathbf{k}}\left[\left(\frac{2\pi}{\alpha}\right)^{3/2}\exp\left(-\frac{1}{2\alpha}|\mathbf{k}|^2\right)\right]$$
$$= -\mathrm{i}(2\pi)^{-3/2}\left(\frac{2}{3}\right)^{1/2}\pi^{-3/4}\alpha^{5/4}\left(\frac{2\pi}{\alpha}\right)^{3/2}\frac{1}{2\alpha}\exp\left(-\frac{1}{2\alpha}|\mathbf{k}|^2-\mathrm{i}\mathbf{k}\cdot\mathbf{x}_c\right)\nabla_{\mathbf{k}}|\mathbf{k}|^2$$
$$= -\mathrm{i}\left(\frac{2}{3}\right)^{1/2}\pi^{-3/4}\alpha^{-5/4}\mathbf{k}\exp\left(-\frac{1}{2\alpha}|\mathbf{k}|^2-\mathrm{i}\mathbf{k}\cdot\mathbf{x}_c\right). \tag{17}$$

Note that with this choice of the Fourier transform we also have

$$\|\tilde{\Psi}(\alpha, \mathbf{k}, \mathbf{x}_c)\|_2 = 1. \tag{18}$$

(iv) To prove the sifting property we split the integration domain into two parts, i.e.,

$$\lim_{\alpha\to\infty}\int_{\mathbf{x}\in\mathbb{R}^3} f(\mathbf{x})\,|\Psi(\alpha, \mathbf{x}, \mathbf{x}_c)|^2\,\mathrm{d}\mathbf{x}$$
$$= \lim_{\alpha\to\infty}\left[\int_{\mathbf{x}\in\mathbb{R}^3\setminus V(\delta)} f(\mathbf{x})\,|\Psi(\alpha, \mathbf{x}, \mathbf{x}_c)|^2\,\mathrm{d}\mathbf{x} + \int_{\mathbf{x}\in V(\delta)} f(\mathbf{x})\,|\Psi(\alpha, \mathbf{x}, \mathbf{x}_c)|^2\,\mathrm{d}\mathbf{x}\right]$$
$$= \lim_{\alpha\to\infty}\int_{\mathbf{x}\in\mathbb{R}^3\setminus V(\delta)} f(\mathbf{x})\,|\Psi(\alpha, \mathbf{x}, \mathbf{x}_c)|^2\,\mathrm{d}\mathbf{x} + \lim_{\alpha\to\infty}\int_{\mathbf{x}\in V(\delta)} f(\mathbf{x})\,|\Psi(\alpha, \mathbf{x}, \mathbf{x}_c)|^2\,\mathrm{d}\mathbf{x}, \tag{19}$$

where $V(\delta)$ is some volume surrounding the point $\mathbf{x}_c$. Now we shall choose $V(\delta)$ in such a way that the first (outer) term in the expression above gives zero. We start by considering a general case where $\mathbf{x} \in V(\delta)$, if $|\mathbf{x} - \mathbf{x}_c| \leqslant \delta$, and δ is some function of α. Then,

$$\lim_{\alpha\to\infty}\int_{\mathbf{x}\in\mathbb{R}^3\setminus V(\delta)} f(\mathbf{x})\,|\Psi(\alpha, \mathbf{x}, \mathbf{x}_c)|^2\,\mathrm{d}\mathbf{x}$$
$$\leqslant \max_{\mathbf{x}\in\mathbb{R}^3}|f(\mathbf{x})|\lim_{\alpha\to\infty}\int_{\mathbf{x}\in\mathbb{R}^3\setminus V(\delta)}|\Psi(\alpha, \mathbf{x}, \mathbf{x}_c)|^2\,\mathrm{d}\mathbf{x} \tag{20}$$
$$= \max_{\mathbf{x}\in\mathbb{R}^3}|f(\mathbf{x})|\lim_{\alpha\to\infty}\frac{2}{3}\pi^{-3/2}\alpha^{5/2}\int_{r=\delta}^{\infty}\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} r^4\,\mathrm{e}^{-\alpha r^2}\sin\theta\,\mathrm{d}\varphi\,\mathrm{d}\theta\,\mathrm{d}r$$
$$= \frac{8\pi}{3}\pi^{-3/2}\max_{\mathbf{x}\in\mathbb{R}^3}|f(\mathbf{x})|\lim_{\alpha\to\infty}\alpha^{5/2}\int_{r=\delta}^{\infty} r^4\,\mathrm{e}^{-\alpha r^2}\,\mathrm{d}r. \tag{21}$$

Successive integration by parts gives

$$\int_{r=\delta}^{\infty} r^4\,\mathrm{e}^{-\alpha r^2}\,\mathrm{d}r = \frac{3}{4\alpha^2}\int_{r=\delta}^{\infty}\mathrm{e}^{-\alpha r^2}\,\mathrm{d}r + \left(\frac{\delta^3}{2\alpha}+\frac{3\delta}{4\alpha^2}\right)\mathrm{e}^{-\alpha\delta^2}. \tag{22}$$

Using this result we continue to analyse the upper bound of (20) as follows:

$$\lim_{\alpha\to\infty}\int_{\mathbf{x}\in\mathbb{R}^3\setminus V(\delta)}|\Psi(\alpha, \mathbf{x}, \mathbf{x}_c)|^2\,\mathrm{d}\mathbf{x} = \lim_{\alpha\to\infty}\frac{8\pi}{3}\pi^{-3/2}\alpha^{5/2}\int_{r=\delta}^{\infty} r^4\,\mathrm{e}^{-\alpha r^2}\,\mathrm{d}r$$
$$= \frac{8\pi}{3}\pi^{-3/2}\lim_{\alpha\to\infty}\alpha^{5/2}\left[\frac{3}{4\alpha^2}\int_{r=\delta}^{\infty}\mathrm{e}^{-\alpha r^2}\,\mathrm{d}r + \left(\frac{\delta^3}{2\alpha}+\frac{3\delta}{4\alpha^2}\right)\mathrm{e}^{-\alpha\delta^2}\right]$$
$$= \frac{2}{\sqrt{\pi}}\lim_{\alpha\to\infty}\alpha^{1/2}\int_{r=\delta}^{\infty}\mathrm{e}^{-\alpha r^2}\,\mathrm{d}r + \lim_{\alpha\to\infty}\frac{8}{3\sqrt{\pi}}\left(\frac{1}{2}\delta^3\alpha^{3/2}+\frac{3}{4}\delta\alpha^{1/2}\right)\mathrm{e}^{-\alpha\delta^2}. \tag{23}$$

Now, choosing, for example, $\delta = \alpha^{-1/3}$, we arrive at

$$\lim_{\alpha\to\infty} \int_{\mathbf{x}\in\mathbb{R}^3\setminus V(\delta)} |\Psi(\alpha, \mathbf{x}, \mathbf{x}_c)|^2 \,d\mathbf{x}$$

$$= \frac{2}{\sqrt{\pi}} \lim_{\alpha\to\infty} \int_{r\sqrt{\alpha}=\alpha^{1/6}}^{\infty} e^{-(r\sqrt{\alpha})^2} \,d(r\sqrt{\alpha}) + \lim_{\alpha\to\infty} \frac{8}{3\sqrt{\pi}} \left(\frac{1}{2}\alpha^{1/2} + \frac{3}{4}\alpha^{1/6}\right) e^{-\alpha^{1/3}}$$

$$\leqslant \frac{2}{\sqrt{\pi}} \lim_{\alpha\to\infty} \int_{z=\alpha^{1/6}}^{\infty} e^{-z} \,dz = \frac{2}{\sqrt{\pi}} \lim_{\alpha\to\infty} e^{-\alpha^{1/6}} = 0. \tag{24}$$

Hence, with this particular choice of $V(\delta)$ the first (outer) term in (19) is zero. Now we shall use the same $V(\delta)$ in the second (inner) term. Taking into account that $f(\mathbf{x})$ is a continuous function, and that with our choice of δ the integration volume $V(\delta)$ tends to the point $\mathbf{x}_c$, we can apply the mean-value theorem, i.e.,

$$\lim_{\alpha\to\infty} \int_{\mathbf{x}\in V(\delta)} f(\mathbf{x}) \,|\Psi(\alpha, \mathbf{x}, \mathbf{x}_c)|^2 \,d\mathbf{x} = \lim_{\alpha\to\infty} f(\mathbf{x}_\alpha) \int_{\mathbf{x}\in V(\delta)} |\Psi(\alpha, \mathbf{x}, \mathbf{x}_c)|^2 \,d\mathbf{x}$$

$$= f(\mathbf{x}_c) \lim_{\alpha\to\infty} \int_{\mathbf{x}\in V(\delta)} |\Psi(\alpha, \mathbf{x}, \mathbf{x}_c)|^2 \,d\mathbf{x}, \tag{25}$$

where $\mathbf{x}_\alpha \in V(\delta)$, and $\mathbf{x}_\alpha \to \mathbf{x}_c$ as $\alpha \to \infty$. Thus, to prove the sifting property it remains to show that

$$\lim_{\alpha\to\infty} \int_{\mathbf{x}\in V(\delta)} |\Psi(\alpha, \mathbf{x}, \mathbf{x}_c)|^2 \,d\mathbf{x} = \frac{8}{3\sqrt{\pi}} \lim_{\alpha\to\infty} \alpha^{5/2} \int_{r=0}^{\alpha^{-1/3}} r^4 \,e^{-\alpha r^2} \,dr$$

$$= \frac{8}{3\sqrt{\pi}} \lim_{\alpha\to\infty} \int_{r\sqrt{\alpha}=0}^{\alpha^{1/6}} (r\sqrt{\alpha})^4 \,e^{-(r\sqrt{\alpha})^2} \,d(r\sqrt{\alpha})$$

$$= \frac{8}{3\sqrt{\pi}} \lim_{\alpha\to\infty} \int_{z=0}^{\alpha^{1/6}} z^4 \,e^{-z^2} \,dz$$

$$= \frac{8}{3\sqrt{\pi}} \int_{z=0}^{\infty} z^4 \,e^{-z^2} \,dz = 1. \tag{26}$$

Finally, we remark that the above proof holds with any $\delta = \alpha^{-1/m}$, where $m \geqslant 3$.

(v) We prove the orthogonality property by considering the absolute value of the dot-product, i.e.,

$$\left|\lim_{\alpha\to\infty} \langle \mathbf{V}(\mathbf{x}), \Psi(\alpha, \mathbf{x}, \mathbf{x}_c)\rangle\right|$$

$$= \left|\lim_{\alpha\to\infty} \sqrt{\frac{2}{3}}\pi^{-3/4}\alpha^{5/4} \int_{r=0}^{\infty}\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} \mathbf{\Theta}^{\mathrm{T}}(\theta,\varphi)\mathbf{V}(r,\theta,\varphi) r^3 e^{-\frac{\alpha}{2}r^2} \sin\theta \,d\varphi \,d\theta \,dr\right|$$

$$\leqslant C \sup_{\mathbf{x}\in\mathbb{R}^3} |\mathbf{\Theta}^{\mathrm{T}}(\theta,\varphi)\mathbf{V}(\mathbf{x})| \lim_{\alpha\to\infty} \alpha^{5/4} \int_{r=0}^{\infty} r^3 e^{-\frac{\alpha}{2}r^2} \,dr$$

$$= C \sup_{\mathbf{x}\in\mathbb{R}^3} |\mathbf{\Theta}^{\mathrm{T}}(\theta,\varphi)\mathbf{V}(\mathbf{x})| \lim_{\alpha\to\infty} \alpha^{5/4} \frac{1}{2(\alpha/2)^2}$$

$$= 2C \sup_{\mathbf{x}\in\mathbb{R}^3} |\mathbf{\Theta}^{\mathrm{T}}(\theta,\varphi)\mathbf{V}(\mathbf{x})| \lim_{\alpha\to\infty} \alpha^{-3/4} = 0, \tag{27}$$

where $0 < C < \infty$, $\mathbf{\Theta} = (\mathbf{x} - \mathbf{x}_c)/|\mathbf{x} - \mathbf{x}_c|$, and the following standard integral was used:

$$\int_0^{\infty} r^{2n+1} \,e^{-pr^2} \,dr = \frac{n!}{2p^{n+1}}, \qquad p > 0, \qquad n = 0, 1, 2, \ldots \tag{28}$$

(vi) In the case of a weakly singular integral operator with finite spatial support D we proceed as follows:

$$\begin{aligned}
&\lim_{\alpha\to\infty}\left\|\int_{\mathbf{x}\in D}\frac{\mathbb{K}(\mathbf{x},\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^{\beta}}\Psi(\alpha,\mathbf{x},\mathbf{x}_{\rm c})\,{\rm d}\mathbf{x}\right\|_2^2\\
&\quad=\lim_{\alpha\to\infty}\int_{\mathbf{x}'\in D}\left|\int_{\mathbf{x}\in D}\frac{\mathbb{K}(\mathbf{x},\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^{\beta}}\Psi(\alpha,\mathbf{x},\mathbf{x}_{\rm c})\,{\rm d}\mathbf{x}\right|^2{\rm d}\mathbf{x}'\\
&\quad=\lim_{\alpha\to\infty}\int_{\mathbf{x}'\in D}\left|\int_{\mathbf{x}\in D}\frac{\mathbb{K}(\mathbf{x},\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^{\beta}}[\Psi_2(\alpha,\mathbf{x},\mathbf{x}_{\rm c})+\Psi_1(\alpha,\mathbf{x},\mathbf{x}_{\rm c})]\,{\rm d}\mathbf{x}\right|^2{\rm d}\mathbf{x}'\\
&\quad\leqslant\lim_{\alpha\to\infty}\int_{\mathbf{x}'\in D}\left|\int_{\mathbf{x}\in D}\frac{\mathbb{K}(\mathbf{x},\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^{\beta}}\Psi_1(\alpha,\mathbf{x},\mathbf{x}_{\rm c})\,{\rm d}\mathbf{x}\right|^2{\rm d}\mathbf{x}'\\
&\qquad+\lim_{\alpha\to\infty}\int_{\mathbf{x}'\in D}\left|\int_{\mathbf{x}\in D}\frac{\mathbb{K}(\mathbf{x},\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^{\beta}}\Psi_2(\alpha,\mathbf{x},\mathbf{x}_{\rm c})\,{\rm d}\mathbf{x}\right|^2{\rm d}\mathbf{x}',
\end{aligned}\tag{29}$$

where the original function Ψ is split into two complementary parts with respect to a small volume around $\mathbf{x}_{\rm c}$ in such a way that

$$\begin{aligned}
&\Psi(\alpha,\mathbf{x},\mathbf{x}_{\rm c})=\Psi_1(\alpha,\mathbf{x},\mathbf{x}_{\rm c})+\Psi_2(\alpha,\mathbf{x},\mathbf{x}_{\rm c}),\\
&\Psi_1(\alpha,\mathbf{x},\mathbf{x}_{\rm c})=0,\qquad \mathbf{x}\in\mathbb{R}^3\backslash V(\delta),\\
&\Psi_2(\alpha,\mathbf{x},\mathbf{x}_{\rm c})=0,\qquad \mathbf{x}\in V(\delta).
\end{aligned}\tag{30}$$

The last integral in (29) is estimated like this

$$\begin{aligned}
&\lim_{\alpha\to\infty}\int_{\mathbf{x}'\in D}\left|\int_{\mathbf{x}\in D}\frac{\mathbb{K}(\mathbf{x},\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^{\beta}}\Psi_2(\alpha,\mathbf{x},\mathbf{x}_{\rm c})\,{\rm d}\mathbf{x}\right|^2{\rm d}\mathbf{x}'\\
&\quad=\lim_{\alpha\to\infty}\|K\Psi_2\|^2\leqslant\|K\|^2\lim_{\alpha\to\infty}\|\Psi_2\|^2=0,
\end{aligned}\tag{31}$$

where we have used the fact that the norm of a weakly singular operator on D is bounded, and the previously derived property (23)–(24). This means that we take $\delta=\alpha^{-1/m}$, $m\geqslant 3$. The remaining integral in (29) requires considerably more work. We shall split the domain of integration over $\mathbf{x}'$ into two parts using yet another small volume $V(\delta')$ surrounding the point $\mathbf{x}_{\rm c}$. Then, taking (30) into account, we obtain

$$\begin{aligned}
&\lim_{\alpha\to\infty}\int_{\mathbf{x}'\in D}\left|\int_{\mathbf{x}\in D}\frac{\mathbb{K}(\mathbf{x},\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^{\beta}}\Psi_1(\alpha,\mathbf{x},\mathbf{x}_{\rm c})\,{\rm d}\mathbf{x}\right|^2{\rm d}\mathbf{x}'\\
&\quad=\lim_{\alpha\to\infty}\int_{\mathbf{x}'\in D\backslash V(\delta')}\left|\int_{\mathbf{x}\in V(\delta)}\frac{\mathbb{K}(\mathbf{x},\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^{\beta}}\Psi_1(\alpha,\mathbf{x},\mathbf{x}_{\rm c})\,{\rm d}\mathbf{x}\right|^2{\rm d}\mathbf{x}'\\
&\qquad+\lim_{\alpha\to\infty}\int_{\mathbf{x}'\in V(\delta')}\left|\int_{\mathbf{x}\in V(\delta)}\frac{\mathbb{K}(\mathbf{x},\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^{\beta}}\Psi_1(\alpha,\mathbf{x},\mathbf{x}_{\rm c})\,{\rm d}\mathbf{x}\right|^2{\rm d}\mathbf{x}'.
\end{aligned}\tag{32}$$

Proceeding with the first of the above integrals we apply the Cauchy–Schwartz inequality and arrive at

$$\begin{aligned}
&\lim_{\alpha\to\infty}\int_{\mathbf{x}'\in D\backslash V(\delta')}\left|\int_{\mathbf{x}\in V(\delta)}\frac{\mathbb{K}(\mathbf{x},\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^{\beta}}\Psi_1(\alpha,\mathbf{x},\mathbf{x}_{\rm c})\,{\rm d}\mathbf{x}\right|^2{\rm d}\mathbf{x}'\\
&\quad\leqslant\lim_{\alpha\to\infty}\int_{\mathbf{x}'\in D\backslash V(\delta')}\int_{\mathbf{x}\in V(\delta)}\frac{\left|\mathbb{K}(\mathbf{x},\mathbf{x}')\boldsymbol{\Theta}\right|^2}{|\mathbf{x}-\mathbf{x}'|^{2\beta}}\,{\rm d}\mathbf{x}\int_{\mathbf{x}\in V(\delta)}|\Psi_1(\alpha,\mathbf{x},\mathbf{x}_{\rm c})|^2\,{\rm d}\mathbf{x}\,{\rm d}\mathbf{x}'\\
&\quad\leqslant\sup_{\mathbf{x},\mathbf{x}'\in D}|\mathbb{K}(\mathbf{x},\mathbf{x}')\boldsymbol{\Theta}|^2\times\lim_{\alpha\to\infty}\int_{\mathbf{x}'\in D\backslash V(\delta')}\\
&\qquad\times\int_{\mathbf{x}\in V(\delta)}\frac{1}{|\mathbf{x}-\mathbf{x}'|^{2\beta}}\,{\rm d}\mathbf{x}\,{\rm d}\mathbf{x}'\int_{\mathbf{x}\in V(\delta)}|\Psi_1(\alpha,\mathbf{x},\mathbf{x}_{\rm c})|^2\,{\rm d}\mathbf{x}.
\end{aligned}\tag{33}$$

Now, if we choose the spherical volume $V(\delta')$ with radius δ' to be larger than the spherical volume $V(\delta)$ with radius δ, then for $\mathbf{x} \in V(\delta)$ and $\mathbf{x}' \in D\backslash V(\delta')$, the distance factor $|\mathbf{x} - \mathbf{x}'|$ will be bounded from below by the difference of the radii of the two volumes. In estimate (31) we have used $\delta = \alpha^{-1/m}, m \geqslant 3$. Hence, for sufficiently large α we can choose $\delta' = \delta^{1/n} = \alpha^{-1/(nm)}$ with $n \geqslant 2$. In this case we have

$$\begin{aligned}
&\lim_{\alpha\to\infty} \int_{\mathbf{x}'\in D\backslash V(\delta')} \int_{\mathbf{x}\in V(\delta)} \frac{1}{|\mathbf{x}-\mathbf{x}'|^{2\beta}}\,\mathrm{d}\mathbf{x} \int_{\mathbf{x}\in V(\delta)} |\Psi_1(\alpha,\mathbf{x},\mathbf{x}_\mathrm{c})|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{x}' \\
&\quad\leqslant \lim_{\alpha\to\infty} \sup_{\mathbf{x}\in V(\delta),\mathbf{x}'\in D\backslash V(\delta')} \frac{1}{|\mathbf{x}-\mathbf{x}'|^{2\beta}} \int_{\mathbf{x}'\in D\backslash V(\delta')} \int_{\mathbf{x}\in V(\delta)} \mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{x}' \\
&\quad\quad\times \int_{\mathbf{x}\in V(\delta)} |\Psi_1(\alpha,\mathbf{x},\mathbf{x}_\mathrm{c})|^2\,\mathrm{d}\mathbf{x} \\
&\quad\leqslant C \lim_{\alpha\to\infty} \frac{\delta^3}{(\delta'-\delta)^{2\beta}} = C \lim_{\alpha\to\infty} \frac{\delta^3}{(\delta^{1/n}-\delta)^{2\beta}} = C \lim_{\alpha\to\infty} \frac{\delta^{3-1/n}}{(1-\delta^{1-1/n})^{2\beta}} \\
&\quad= C \lim_{\alpha\to\infty} \frac{\alpha^{(1-3n)/(nm)}}{(1-\alpha^{(1-n)/(nm)})^{2\beta}} = 0,
\end{aligned} \tag{34}$$

i.e., the first of the two integrals in (32) is zero. Applying the Cauchy–Schwartz inequality we estimate the last integral in (32) as follows:

$$\begin{aligned}
&\lim_{\alpha\to\infty} \int_{\mathbf{x}'\in V(\delta')} \left| \int_{\mathbf{x}\in V(\delta)} \frac{\mathbb{K}(\mathbf{x},\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^{\beta}} \Psi_1(\alpha,\mathbf{x},\mathbf{x}_\mathrm{c})\,\mathrm{d}\mathbf{x} \right|^2 \mathrm{d}\mathbf{x}' \\
&\quad\leqslant K \lim_{\alpha\to\infty} \int_{\mathbf{x}'\in V(\delta')} \int_{\mathbf{x}\in V(\delta)} \frac{1}{|\mathbf{x}-\mathbf{x}'|^{\beta}}\,\mathrm{d}\mathbf{x} \int_{\mathbf{x}\in V(\delta)} \frac{|\Psi_1(\alpha,\mathbf{x},\mathbf{x}_\mathrm{c})|^2}{|\mathbf{x}-\mathbf{x}'|^{\beta}}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{x}' \\
&\quad\leqslant L \lim_{\alpha\to\infty} (\delta')^{3-\beta} \int_{\mathbf{x}'\in V(\delta')} \int_{\mathbf{x}\in V(\delta)} \frac{|\Psi_1(\alpha,\mathbf{x},\mathbf{x}_\mathrm{c})|^2}{|\mathbf{x}-\mathbf{x}'|^{\beta}}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{x}' \\
&\quad\leqslant M \lim_{\alpha\to\infty} (\delta')^{2(3-\beta)} \int_{\mathbf{x}\in V(\delta)} |\Psi_1(\alpha,\mathbf{x},\mathbf{x}_\mathrm{c})|^2\,\mathrm{d}\mathbf{x} \\
&\quad= N \lim_{\alpha\to\infty} (\delta')^{6-2\beta} = N \lim_{\alpha\to\infty} \alpha^{-(6-2\beta)/(nm)} = 0,
\end{aligned} \tag{35}$$

where $\delta' = \alpha^{-1/(nm)}$, $m \geqslant 3$, $n \geqslant 2$, while $0 < \beta < 3$ by the conditions of the theorem. Thus, we have shown that (29) is, indeed, zero. This completes the proof of the theorem. □

3. Electromagnetic singular modes

Consider the volume integral equation of electromagnetic scattering on a nonmagnetic object of finite spatial extent D:

$$\mathbf{E}^{\mathrm{in}}(\mathbf{x},\omega) = \mathbf{E}(\mathbf{x},\omega) - \left[k_0^2(\omega) + \nabla\nabla\cdot\right] \int_{\mathbf{x}'\in D} g(\mathbf{x}-\mathbf{x}',\omega)\chi(\mathbf{x}',\omega)\mathbf{E}(\mathbf{x}',\omega)\,\mathrm{d}V, \tag{36}$$

where $\mathbf{E}^{\mathrm{in}}$ and $\mathbf{E}$ are the incident and total electric fields, correspondingly. This equation is obtained directly from the frequency-domain Maxwell's equations and takes into account the radiation condition at infinity in a most natural form. The medium parameters are contained in the contrast function χ, which in terms of the complex permittivity function ε will look like

$$\chi(\mathbf{x},\omega) = \frac{\varepsilon(\mathbf{x},\omega)}{\varepsilon_0} - 1 = \varepsilon_\mathrm{r}(\mathbf{x},\omega) - 1. \tag{37}$$

The vacuum wavenumber is $k_0 = \omega/c$, and the scalar Green's function is given by

$$g(\mathbf{x},\omega) = \frac{\mathrm{e}^{\mathrm{i}k_0|\mathbf{x}|}}{4\pi|\mathbf{x}|}. \tag{38}$$

Carrying out the two spatial derivatives we arrive at the following singular integral equation:

$$\mathbf{E}^{\text{in}}(\mathbf{x},\omega) = \left[1 + \frac{1}{3}\chi(\mathbf{x},\omega)\right]\mathbf{E}(\mathbf{x},\omega) - \lim_{\delta\to 0}\int_{\mathbf{x}'\in D\setminus|\mathbf{x}-\mathbf{x}'|<\delta} \mathbb{G}_0(\mathbf{x}-\mathbf{x}')\chi(\mathbf{x}',\omega)\mathbf{E}(\mathbf{x}',\omega)\,\mathrm{d}\mathbf{x}' - \int_{\mathbf{x}'\in D} \mathbb{G}_1(\mathbf{x}-\mathbf{x}',\omega)\chi(\mathbf{x}',\omega)\mathbf{E}(\mathbf{x}',\omega)\,\mathrm{d}\mathbf{x}'. \tag{39}$$

For the definitions of the Green tensors $\mathbb{G}_0$ and $\mathbb{G}_1$ we refer to our previous publication on this subject [7].

Theorem 2. *The vector-valued function* $\Psi(\alpha,\mathbf{x},\mathbf{x}_c)$ *defined in theorem 1 generates the essential mode of the electromagnetic field corresponding to the point of essential spectrum*

$$\lambda_{\text{ess}} = \varepsilon_r(\mathbf{x}_c,\omega). \tag{40}$$

Proof. Since we have already established the normalization (i) and the singularity of the sequence (ii) in theorem 1, we only need to prove the following analogue of (2),

$$\begin{aligned} V = \lim_{\alpha\to\infty} \Bigg\| \left[1 + \frac{1}{3}\chi(\mathbf{x},\omega)\right]\Psi(\alpha,\mathbf{x},\mathbf{x}_c) &- \lim_{\delta\to 0}\int_{\mathbf{x}'\in D\setminus|\mathbf{x}-\mathbf{x}'|<\delta} \mathbb{G}_0(\mathbf{x}-\mathbf{x}')\chi(\mathbf{x}',\omega)\Psi(\alpha,\mathbf{x}',\mathbf{x}_c)\,\mathrm{d}\mathbf{x}' \\ &- \int_{\mathbf{x}'\in D} \mathbb{G}_1(\mathbf{x}-\mathbf{x}',\omega)\chi(\mathbf{x}',\omega)\Psi(\alpha,\mathbf{x}',\mathbf{x}_c)\,\mathrm{d}\mathbf{x}' - \lambda\Psi(\alpha,\mathbf{x},\mathbf{x}_c)\Bigg\|_D^2 \\ = \lim_{\alpha\to\infty} \Bigg\| \left[\frac{2}{3} + \frac{1}{3}\varepsilon_r(\mathbf{x}) - \lambda\right]\Psi(\alpha,\mathbf{x},\mathbf{x}_c) &- \lim_{\delta\to 0}\int_{\mathbf{x}'\in D\setminus V(\delta)} \mathbb{G}_0(\mathbf{x}-\mathbf{x}')\left[\varepsilon_r(\mathbf{x}') - \varepsilon_r(\mathbf{x}) + \varepsilon_r(\mathbf{x}) - 1\right]\Psi(\alpha,\mathbf{x}',\mathbf{x}_c)\,\mathrm{d}\mathbf{x}' \\ &- \int_{\mathbf{x}'\in D} \mathbb{G}_1(\mathbf{x}-\mathbf{x}',\omega)\left[\varepsilon_r(\mathbf{x}') - 1\right]\Psi(\alpha,\mathbf{x}',\mathbf{x}_c)\,\mathrm{d}\mathbf{x}'\Bigg\|_D^2 = 0, \end{aligned} \tag{41}$$

for $\lambda = \varepsilon_r(\mathbf{x}_c,\omega)$. The L_2 norm is taken over the finite spatial support D. First, we rearrange (41) and decompose it into separate terms:

$$\begin{aligned} V \leqslant{}& \lim_{\alpha\to\infty} \|[\varepsilon_r(\mathbf{x}) - \lambda]\,\Psi(\alpha,\mathbf{x},\mathbf{x}_c)\|_D^2 \\ &+ \lim_{\alpha\to\infty} \left\|[1-\varepsilon_r(\mathbf{x})]\left[\frac{2}{3}\Psi(\alpha,\mathbf{x},\mathbf{x}_c) + \lim_{\delta\to 0}\int_{\mathbf{x}'\in D\setminus V(\delta)} \mathbb{G}_0(\mathbf{x}-\mathbf{x}')\Psi(\alpha,\mathbf{x}',\mathbf{x}_c)\,\mathrm{d}\mathbf{x}'\right]\right\|_D^2 \\ &+ \lim_{\alpha\to\infty} \Bigg\| \lim_{\delta\to 0}\int_{\mathbf{x}'\in D\setminus V(\delta)} \mathbb{G}_0(\mathbf{x}-\mathbf{x}')[\varepsilon(\mathbf{x},\omega) - \varepsilon(\mathbf{x}',\omega)]\Psi(\alpha,\mathbf{x}',\mathbf{x}_c)\,\mathrm{d}\mathbf{x}' \\ &- \int_{\mathbf{x}'\in D} \mathbb{G}_1(\mathbf{x}-\mathbf{x}',\omega)[\varepsilon_r(\mathbf{x}') - 1]\Psi(\alpha,\mathbf{x}',\mathbf{x}_c)\,\mathrm{d}\mathbf{x}'\Bigg\|_D^2. \end{aligned} \tag{42}$$

With $\varepsilon_r(\mathbf{x},\omega)$ Hölder-continuous in $\mathbb{R}^3$ all integral operators in the last term are weakly singular. Hence, from property (vi) of theorem 1 the last term in (42) is zero.

From the second term in (42) we only consider the expression in the large square brackets, which we extend to $\mathbb{R}^3$. Denoting by $\mathcal{F}\{\cdots\}$ and $\mathcal{F}^{-1}\{\cdots\}$ the forward and inverse three-dimensional Fourier transforms, we obtain

$$\begin{aligned}
&\frac{2}{3}\Psi(\alpha,\mathbf{x},\mathbf{x}_c)+\lim_{\delta\to 0}\int_{\mathbf{x}'\in\mathbb{R}^3\setminus V(\delta)}\mathbb{G}_0(\mathbf{x}-\mathbf{x}')\Psi(\alpha,\mathbf{x}',\mathbf{x}_c)\,\mathrm{d}\mathbf{x}'\\
&\quad=\mathcal{F}^{-1}\left\{\frac{2}{3}\mathcal{F}\{\Psi(\alpha,\mathbf{x},\mathbf{x}_c)\}+\mathcal{F}\left\{\lim_{\delta\to 0}\int_{\mathbf{x}'\in\mathbb{R}^3\setminus V(\delta)}\mathbb{G}_0(\mathbf{x}-\mathbf{x}')\Psi(\alpha,\mathbf{x}',\mathbf{x}_c)\,\mathrm{d}\mathbf{x}'\right\}\right\}\\
&\quad=\mathcal{F}^{-1}\left\{\frac{2}{3}\tilde{\Psi}(\alpha,\mathbf{k},\mathbf{x}_c)+\left[\frac{1}{3}\mathbb{I}-\tilde{\mathbb{Q}}\right]\tilde{\Psi}(\alpha,\mathbf{k},\mathbf{x}_c)\right\}\\
&\quad=\mathcal{F}^{-1}\left\{\tilde{\Psi}(\alpha,\mathbf{k},\mathbf{x}_c)-\frac{\mathbf{k}\mathbf{k}^{\mathrm{T}}}{|\mathbf{k}|^2}\tilde{\Psi}(\alpha,\mathbf{k},\mathbf{x}_c)\right\}=0,
\end{aligned}\qquad (43)$$

where we have used the previously derived result about the Fourier transform of a simple singular operator [7], the explicit form of $\tilde{\Psi}(\alpha,\mathbf{k},\mathbf{x}_c)$, see property (iii), and the fact that $\mathbf{k}^{\mathrm{T}}\mathbf{k}=|\mathbf{k}|^2$. Hence, the second term in (42) is also zero.

Finally, applying property (iv) of theorem 1 we see that the first term in (42) is zero, if equality (40) holds. □

4. Conclusions

We have shown that the electromagnetic essential (singular) mode corresponding to the point $\lambda_{\mathrm{ess}}=\varepsilon(\mathbf{x},\omega)/\varepsilon_0$ of essential spectrum is, in fact, the square root of the delta function located at $\mathbf{x}$. The unique mathematical properties of the latter are summarized in theorem 1. Hence, the essential resonance, which may occur whenever the dielectric permittivity of an object gets close to zero, leads to the excitation of this extremely localized mode. This happens in plasmas, metamaterials and metals. The mathematical structure of essential modes is unusual and, to our best knowledge, has never been analysed before. In particular, it is not clear how exactly the energy is transferred (if it *is* transferred) between the normal modes of the point spectrum and the essential modes. Elsewhere, we show that it is impossible to associate any causal evolution with this process, and there may be a jump-like chaotic transformation [20].

Another conclusion stems from the observation that singular modes do not belong to the Hilbert space (property (ii) of theorem 1), where all 'proper' solutions of the Maxwell equations live. Hence, one may consider the excitation of singular modes as some kind of 'deflation' of the electromagnetic field. Recall that, due to the normalization property (i) of theorem 1, the electromagnetic energy associated with a singular mode is well defined. Such a disappearance of a well-defined portion of energy from the Hilbert-space part of the electromagnetic field may be an interesting alternative model of the electromagnetic absorption.

It is also important to note the direct relation of the electromagnetic essential spectrum and its singular modes to the pseudospectrum and wave-packet pseudomodes [16], which was very helpful in our work. There is an obvious similarity of the Weyl definition (2) and the definition of the pseudospectrum, where instead of zero one should simply put a small ϵ on the right-hand side of (2). Subsequently, we arrive at two distinct possibilities. The first is where $n\to\infty$, i.e., in our case $\alpha\to\infty$. Then, points λ_{ps} satisfying $|\lambda_{\mathrm{ps}}-\varepsilon_{\mathrm{r}}(\mathbf{x},\omega)|\leqslant\epsilon$ will belong to the pseudospectrum, while the corresponding modes will be singular. The second case is where $\lambda_{\mathrm{ps}}=\lambda_{\mathrm{ess}}=\varepsilon_{\mathrm{r}}(\mathbf{x},\omega)$ or very close to it, but $\alpha\leqslant\delta(\epsilon)$. In this case, we stop the sequence of $\Psi(\alpha,\mathbf{x}_c,\mathbf{x})$, at some finite α, for which the norm in (41) equals ϵ. Although it is difficult to derive an explicit relation for $\delta(\epsilon)$, we can anticipate that $\Psi(\delta(\epsilon),\mathbf{x}_c,\mathbf{x})$ will be highly localized in space around the point $\mathbf{x}_c$. In this case the mode is not singular and belongs to the Hilbert space. These two physically distinct possibilities emphasize the non-unique nature of the pseudospectrum as defined in [16] and elsewhere.

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